Thoughts on Rationalizing Algebra in Ways that Serve Kids, Not Universities

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Today's Goal

To provoke and inform your thinking about the need and directions for significantly revising the traditional Algebra I and Algebra II courses to ensure relevance, real rigor and fairness that truly meets the needs of all students.

Today's Agenda

- My premise
- Some perspectives
- Some proposals

My Premise

As currently implemented, high school Algebra 1 and 2 are not working and not meeting either societal or student needs.

As currently implemented, high school Algebra 1 and 2 are not working and not meeting either societal or student needs

- Increasingly obsolete and useless symbol manipulation focus at the expense of functions, models, applications, big ideas and statistics
- An impossible to "cover" scope and sequence
- 1200 page tomes
- 1st half of Algebra 2 = 2 to 1 dilation of Algebra 1
- Extraordinarily high failure rates
- Turns off millions to mathematics
- Designed for a very narrow slice of the cohort
- Neither relevant, rigorous, nor fair!

Some perspectives to support my premise

You choose…

How many of you believe that:

1/2 is simpler than 2/4 ?

Then why isn't

$1/\sqrt{2}$ simpler than $\sqrt{2}/2$?

And who really cares when the important skill is knowing where either $1/\sqrt{2}$ or $\sqrt{2}/2$ live on the number line.

That is: $\sqrt{2} \sim 1.4$ so $1/1.4 \sim 0.7$

And this is rational?

So just for fun…

Simplify:

$$\frac{45}{\sqrt{2} + \sqrt{7}}$$

Actual retail value: 11.08

Versus substance

If $0 < x < 1$, which of the following is greatest?

a) $1/x$

b) $\sqrt{x}$

c) x

d) x^2

- **What if $x > 1$?**
- **What if $x < -1$?**

Alg II L_3 Exam   Do not write on this exam.   25

I Perform the indicated operation and simplify.

1. $\dfrac{9x^7}{36x^{10}y^3}$ a) $\dfrac{3}{12x^3y^3}$ b) $\dfrac{x^7}{4x^{10}y^3}$ c) $4x^3y^3$ d) $\frac{1}{4}x^3y^3$ e) $\dfrac{1}{4x^3y^3}$

2. $\dfrac{x^2-x-12}{x^2+10x+21}$ a) $\dfrac{x-4}{x+7}$ b) $\dfrac{x+4}{x+7}$ c) $\dfrac{1-x-4}{1+10+x}$ d) $\dfrac{4}{7}$ e) $\dfrac{(x+12)(x-1)}{(x+3)(x+7)}$

3. $\dfrac{3x^4}{4a^2y^2} \cdot \dfrac{8a^6y^3}{6x^6}$ a) $\dfrac{a^2y^3x^{-2}}{y^2}$ b) $\dfrac{2a^4y}{2x^2}$ c) $\dfrac{a^4y}{x^2}$ d) $24a^3y^5$ e) $\dfrac{9x^{10}}{16a^3y^5}$

4. $\dfrac{x^2-3x-10}{x^2-4} \cdot \dfrac{x-2}{x-5}$ a) 0 b) $\dfrac{x+2}{x-2}$ c) 1 d) $\dfrac{x-5}{x-2}$ e) $\dfrac{x^3-3x^2+20}{x^3+20}$

5. $\dfrac{36x^2y^2}{15b^3z} \div \dfrac{\text{[illegible]}y^4}{5\text{[illegible]}z^3}$ a) $\dfrac{2z^2}{3b^2xy^2}$ b) $\dfrac{3b^2z^2}{6xy^2}$ c) $\dfrac{2b^2z^2}{xy^2}$ d) $\dfrac{2z^2}{b^2xy^2}$ e) $\dfrac{2}{\text{[illegible]}b^2}$

6. $\dfrac{x^2+4x+3}{x^2-y^2} \div \dfrac{x+3}{x-y}$ a) 0 b) $\dfrac{x+1}{x+y}$ c) $\dfrac{x+1}{x-y}$ d) $\dfrac{x-1}{x+y}$ e) $\dfrac{1}{y}$

7. $\dfrac{8x^2-3x+2}{x^3} + \dfrac{x^2+5x-4}{x^3}$ a) $\dfrac{9x^2+2x+2}{x^3}$ b) $\dfrac{8x^4+2x-2}{x^6}$ c) $\dfrac{9x^2+2x-2}{2x^3}$
d) $\dfrac{9x^2+2x-2}{x^3}$ e) $\dfrac{8x^2-8x-6}{x^3}$

8. $\dfrac{4y+3}{y-2} - \dfrac{y-2}{y-2}$ a) $\dfrac{3y+1}{y-2}$ b) $\dfrac{3y+5}{y-2}$ c) 0 d) $\dfrac{5y+5}{y-2}$ e) $\dfrac{5y+1}{y-2}$

9. $\dfrac{5x}{3} + \dfrac{4y}{5}$ a) 0 b) $25x+12y$ c) $\dfrac{37xy}{15}$ d) $\dfrac{5x+4y}{8}$ e) $\dfrac{25x+12y}{15}$

10. $\frac{4}{x} - \frac{5}{y}$ a) -1 b) $\frac{4y-5x}{xy}$ c) $\frac{5x+4y}{xy}$ d) $-\frac{1}{xy}$ e) $\frac{4y-5x}{xy}$

11. $\frac{18x^2+24x^3-36x^4}{6x^2}$ a) $3x^2+4x^3-6x$ b) $12x^2+18x^3-30x^4$ c) $3x^2+4x^3+6x$ d) $12x^2;\ 18x-30$ e) $24x^3+144x^4-216x^6$

12. $x-5 \overline{) x^2-15x+50}$ a) $x-10$ b) $x+10$ c) $x+10 \frac{5}{x-5}$ d) $x-10 \frac{5}{x-5}$ e) $x-45$

II Solve for x

13. $\frac{x}{3} + \frac{x}{4} = 21$ a) 36 b) 6 c) 126 d) $\frac{3}{4}$ e) $83\frac{1}{3}$

14. $\frac{3}{x+1} = \frac{5}{x-3}$ a) b) 1 c) -7 d) $\pm 3\sqrt{3}$ e) 0

15. $\frac{3}{4x} + \frac{1}{10} = \frac{3}{5x}$ a) $-\frac{3}{2}$ b) $\frac{3}{2}$ c) $-\frac{5}{2}$ d) $\frac{2}{3}$ e) 0

III Simplify each of the following

16. 8^0 a) 1 b) 0 c) 5 d) 8 e) not possible

17. $5x^0$ a) 0 b) 5 c) 1 d) $5x$ e) 50

IV Write without negative exponents and simplify

18. $\frac{3xy^{-3}}{15a^{-2}b^2}$ a) $\frac{3xa^2}{15b^2y^3}$ b) $\frac{3x a^2}{15 b^2 y^3}$ c) $\frac{a^2 x}{5 b^2 y^3}$ d) $\frac{\frac{3x}{y^3}}{\frac{15b^2}{a^2}}$ e) None of These

10. $\frac{4}{x} - \frac{5}{y}$ a) -1 b) $\frac{4x-5y}{xy}$ c) $\frac{5x+4y}{xy}$ d) $-\frac{1}{xy}$ e) $\frac{4y-5x}{xy}$

11. $\frac{18x^5+24x^4-36x^3}{6x^2}$ a) $3x^3+4x^2-6x$ b) $12x^3+18x^2-30x$ c) $3x^3+4x^2+6x$ d) $12x^2\ 18x-30$ e) $54x^7+144x^6-216x^5$

12. $x-5\,\overline{)\,x^2-15x+50}$ a) $x-10$ b) $x+10$ c) $x+10\ \frac{5}{x-5}$ d) $x-10\ \frac{-5}{x-5}$ e) $x-45$

II Solve for x

13. $\frac{x}{3}+\frac{x}{4}=21$ a) 36 b) 3 c) 126 d) $\frac{3}{2}$ e) $82\ ^2/_3$

14. $\frac{3}{x+1} = \frac{2}{x-3}$ a) b) 1 c) -4 d) $\pm 3\sqrt{2}$ e) $\emptyset$

15. $\frac{3}{4x} + \frac{1}{10} = \frac{3}{5x}$ a) $-\frac{2}{3}$ b) $\frac{3}{2}$ c) $-\frac{3}{2}$ d) $\frac{2}{3}$ e) 0

III Simplify each of the following

16. 8^0 a) 1 b) 0 c) 2 d) 8 e) not possible

17. $2x^0$ a) 0 b) x c) 1 d) 2x e) 20

IV Write without negative exponents and simplify

18. $\frac{3xy^{-3}}{2a^{-2}b^2}$ a) $\frac{3x-y^3}{2b^2a^2}$ b) $\frac{3a^2x}{2b^2y^3}$ c) $\frac{6a^2x}{3b^2y}$ d) $\frac{\frac{3x}{y^3}}{\frac{2b^2}{a^2}}$ e) none of these

A little synthetic division perhaps?

Or perhaps you would prefer ignoring all of the technological advances of the past 25 years and doing some factoring for fun?

To summarize our expectations….

Simplify	Solve
Factor	Graph

vs.

Find	Express
Display	Model
Represent	Solve
Predict	Demonstrate

Exhibit A

As mathematics colonizes diverse fields, it develops dialects that diverge from the "King's English" of functions, equations, definitions and theorems. These newly important dialects employ the language of search strategies, data structures, confidence intervals and decision trees.

- Steen

Exhibit B

Evidence from a half-century of reform efforts shows that the mainstream tradition of focusing school mathematics on preparation for a calculus-based post-secondary curriculum is not capable of achieving urgent national goals and that no amount of tinkering in likely to change that in any substantial degree.

- Steen

The pipeline Exhibit

1985:	3,800,000	Kindergarten students
1998:	2,810,000	High school graduates
1998:	1,843,000	College freshman
2002:	1,292,000	College graduates
2002:	150,000	STEM majors
2006:	1,200	PhD’s in mathematics

Factoid 1

20 states now require Algebra 1 or its equivalent for graduation and this number increases to 29 in 2014.

Factoid 2

The NAEP High School Transcript Studies reveal that 55.4% of high school graduates in 1982 took no mathematics beyond Algebra 1, but by 2004, this percentage was reduced to 23.3%, meaning that many students who 27 years ago ended their high school program with Algebra 1 are now expected, and in some cases required, to succeed to Geometry and Algebra 2 (NCES, 2007).

Factoid 3

Only 23% of grade 12 students, most of whom had completed courses well beyond Algebra 1, scored at or above proficient on the 2005 NAEP Mathematics Assessment on which 35% of the items are in the algebra domain (National Center for Education Statistics, 2007).

Traditional Algebra 1

Unit 1: Expressions and Equations

Chapter 1	**The Language of Algebra**
Chapter 2	**Real Numbers**
Chapter 3	**Solving Linear Equations**

Unit 2: Linear Functions

Chapter 4	**Graphing Relations and Functions**
Chapter 5	**Analyzing Linear Equations**
Chapter 6	**Solving Linear Inequalities**
Chapter 7	**Solving Systems of Linear Equation and Inequalities**

Unit 3: Polynomials and Nonlinear Functions

Chapter 8	**Polynomials**
Chapter 9	**Factoring**
Chapter 10	**Quadratic and Exponential Functions**

Unit 4: Radical and Rational Functions

Chapter 11	**Radical Expressions and Triangles**
Chapter 12	**Rational Expressions and Equations**

Unit 5: Data Analysis

Chapter 13	**Statistics**
Chapter 14	**Probability**

Traditional Algebra 2

Chapter 1:	Analyzing Equations and Inequalities
Chapter 2:	Graphing Linear Relations and Functions
Chapter 3:	Solving Systems of Linear Equations and Inequalities
Chapter 4:	Using Matrices
Chapter 5:	Exploring Polynomials and Radical Expression
Chapter 6:	Exploring Quadratic Functions and Inequalities
Chapter 7:	Analyzing Conic Sections
Chapter 8:	Exploring Polynomial Functions
Chapter 9:	Exploring Rational Expressions
Chapter 10:	Exploring Exponential and Logarithmic Functions
Chapter 11:	Investigating Sequences and Series
Chapter 12:	Investigating Discrete Mathematics and Probability
Chapter 13:	Exploring Trigonometric Functions
Chapter 14:	Using Trigonometric Graphs and Identities

And what are the outcomes?

Achieve ADP Algebra I 2009 Exam

33,446 students (KY, OH, RI and NJ)

Level	Scale Score	% of Students
Advanced	850-575	1.6%
Proficient	574-450	16.4%
Basic	449-387	26.2%
Below Basic	386-300	55.8%

Ave. Scale Score: 384 (850-300)

And what are the outcomes?

Achieve ADP Algebra II 2009 Exam

102,396 students (13 states, 60% AZ & IN)

Level	Scale Score	% of Students
Well –prepared	1650-1275	3.5%
Prepared	1274-1150	11.1%
Needs preparation	1149-900	85.4%

Ave. Scale Score: 1032   (900 - 1650)

Additional ADP Findings

- On both the Algebra I and Algebra II exams, students earned, on average, only 11% and 14%, respectively, of the possible points available on constructed-response items.
- On the Algebra II exam in both 2008 and 2009, nearly one-third of the students earned no points on the 2-point or 4-point constructed response items.

Yes, Virginia or Houston or whoever,

WE HAVE A PROBLEM!!!

(and no amount of tinkering around the edges is going to fix it)

Yeah, but…..

But Steve…..

- the SAT,
- the ACT,
- the mathematicians,
- the college placement tests….

But friends, what about our students?

Let’s see….

Siti packs her clothes into a suitcase and it weighs 29 kg.

Rahim packs his clothes into an identical suitcase and it weighs 11 kg.

Siti’s clothes are three times as heavy as Rahims.

What is the mass of Rahim’s clothes?

What is the mass of the suitcase?

The old (only) way:

Let S = the weight of Siti's clothes

Let R = the weight of Rahim's clothes

Let X = the weight of the suitcase

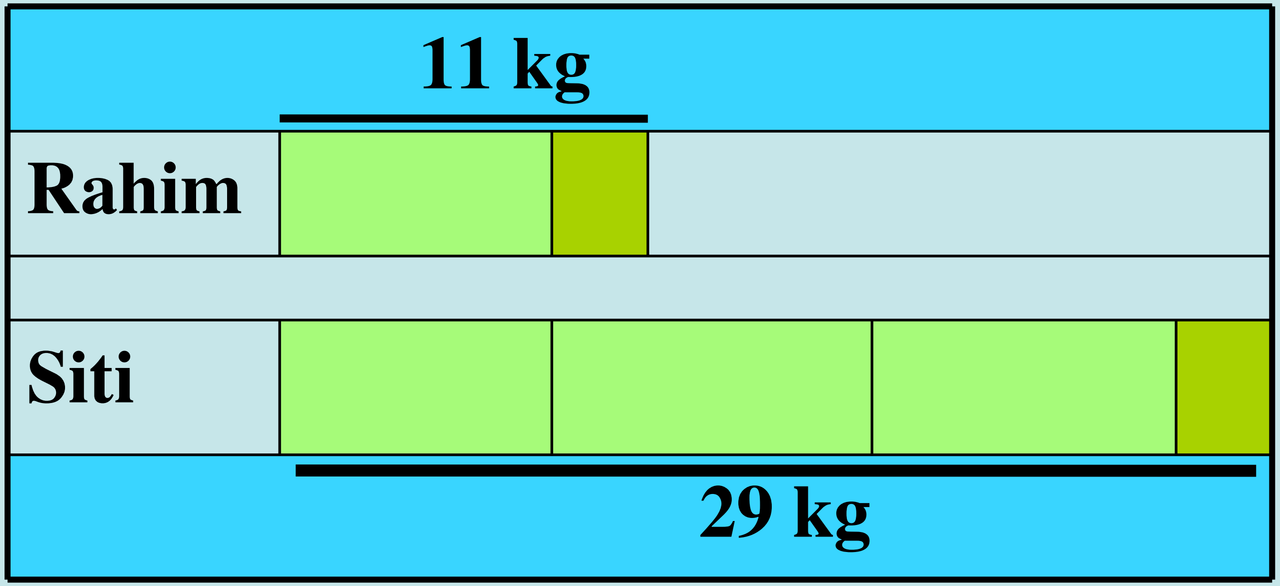
$S = 3R$ $\qquad S + X = 29$ $\qquad R + X = 11$

so by substitution: $3R + X = 29$

and by subtraction: $2R = 18$

so $R = 9$ and $X = 2$

Or using a model:



Thinking vs. Math

24. The number of boys attending Fairfield High School is twice the number of girls.

    If 1/6 of the boys and 1/4 of the girls are in the school band, what fraction of the student are in the school band?

5/36 7/36 2/9 7/24 5/12

The “right” way:

Let B = the number of boys

Let G = the number of girls

So: 2G = B

Find: $\frac{1/6\ B + 1/4\ G}{B + G} = \frac{1/6\ (2G) + 1/4\ G)}{2G + G}$

$= (2/6\ G + 1/4\ G)/3G$

$= 7/12\ G\ /\ 3G\ =\ 7/36$

The Stanley Kaplan Approach

Boys		Girls
2	:	1
1/6		1/4

Try: 100 students – 2 to 1 – no

Try: 90 students – 60 and 30, 6 and oops

Try: 6ths & 4ths – 24 and 48 – 8 + 6 out of 72

14/72 = 7/36

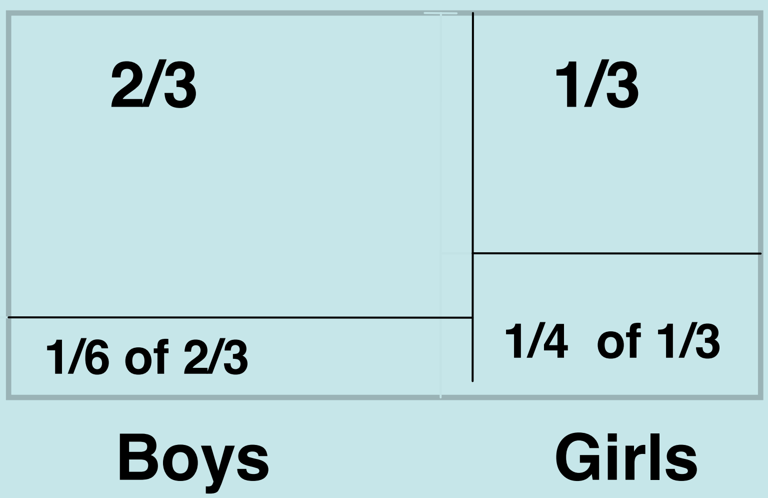
Or even better for some:

- K kids, ergo boys = 2/3K and girls = 1/3K

  Band = 1/6 (2/3K) + 1/4 (1/3K)

  = (1/9 + 1/12) K or 7/36 K

- OR



A glimmer of hope

"Explore the feasibility of offering a mathematics pathway to college for secondary students that is equally rigorous to the calculus pathway and that features deeper study of statistics, data analysis, and related discrete mathematics applications, beginning with a redesigned Algebra II course"

- The Opportunity Equation (June, 2009)

More than a glimmer of hope

Common Core State Standards

College and workplace readiness standards:

- 10 domains
- Concepts
- Coherence
- Skills

CCSSI

- Number
- Quantity
- Expressions
- Equations
- Functions
- Modeling
- Shape
- Coordinate
- Probability
- Statistics

CCSSI – Equation

Equations

Core Concepts   Students understand that:

A. An equation is a statement that two expressions are equal.

B. The solutions of an equation are the values of the variables that make the resulting numerical statement true.

C. The steps in solving an equation are guided by understanding and justified by logical reasoning.

D. Equations not solvable in one number system may have solutions in a larger number system.

Coherent Understanding of Equations.

•An equation is a statement that two expressions are equal. Solutions to an equation are the values of the variables in it that make it true. If the equation is true for all values of the variables, then we call it an identity; identities are often discovered by manipulating one expression into another.

•The solutions of an equation in one variable form a set of numbers; the solutions of an equation in two variables form a set of ordered pairs, which can be graphed in the plane. Equations can be combined into systems to be solved simultaneously.

•An equation can be solved by successively transforming it into one or more simpler equations. The process is governed by deductions based on the properties of equality. For example, one can add the same constant to both sides without changing the solutions, but squaring both sides might lead to extraneous solutions. Strategic competence in solving includes looking ahead for productive manipulations and anticipating the nature and number of solutions.

•Some equations have no solutions in a given number system, stimulating the formation of expanded number systems (integers, rational numbers, real numbers and complex numbers).

•A formula is a type of equation. The same solution techniques used to solve equations can be used to rearrange formulas. For example, the formula for the area of a trapezoid, = 1/2(1+ 2)h, can be solved for *h using the same deductive process.*

•Inequalities can be solved in much the same way as equations. Many, but not all, of the properties of equality extend to the solution of inequalities.

CCSSI - Equations

1 Understand a problem and formulate an equation to solve it.

Extend to inequalities and systems.

2 Solve equations in one variable using manipulations guided by the rules of arithmetic and the properties of equality.

Solve linear equations with procedural fluency. For quadratic equations, include solution by inspection, by factoring, or by using the quadratic formula. Understand that the quadratic formula comes from completing the square. Include simple absolute value equations solvable by direct inspection and by understanding the interpretation of absolute value as distance.

3 Rearrange formulas to isolate a quantity of interest.

Exclude cases that require extraction of roots or inverse functions.

4 Solve systems of equations.

Focus on pairs of simultaneous linear equations in two variables. Include algebraic techniques, graphical techniques and solving by inspection.

5 Solve linear inequalities in one variable and graph the solution set on a number line.

Emphasize solving the associated equality and determining on which side of the solution of the associated equation the solutions to the inequality lie.

6 Graph the solution set of a linear inequality in two variables on the coordinate plane.

CCSSI - Modeling

Core Concepts

Students understand that:

A. Mathematical models involve choices and assumptions that abstract key features from situations to help us solve problems.

B. Even very simple models can be useful.

CCSSI - Modeling

A Coherent Understanding of Modeling.

Modeling uses mathematics to help us make sense of the real world—to understand quantitative relationships, make predictions, and propose solutions.

- **A model can be very simple, such as a geometric shape to describe a physical object like a coin. Even so simple a model involves making choices. It is up to us whether to model the solid nature of the coin with a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. For some purposes, we might even choose to adjust the right circular cylinder to model more closely the way the coin deviates from the cylinder.**
- **In any given situation, the model we devise depends on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models we can create and analyze is constrained as well by the limitations of our mathematical and technical skills. For example, modeling a physical object, a delivery route, a production schedule, or a comparison of loan amortizations each requires different sets of tools. Networks, spreadsheets and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations. One of the insights provided by mathematical modeling is that essentially the same mathematical structure might model seemingly different situations.**
- **The basic modeling cycle is one of (1) identifying the key features of a situation, (2) creating geometric, algebraic or statistical objects that describe key features of the situation, (3) analyzing and performing operations on these objects to draw conclusions and (4) interpreting the results of the mathematics in terms of the original situation. Choices and assumptions are present throughout this cycle.**

CCSSI - Modeling

1 Model numerical situations.

Include readily applying the four basic operations in combination to solve multi-step quantitative problems with dimensioned quantities; making estimates to introduce numbers into a situation and get problems started; recognizing proportional or near-proportional relationships and analyzing them using characteristic rates and ratios.

2 Model physical objects with geometric shapes.

Include common objects that can reasonably be idealized as two- and three-dimensional geometric shapes. Identify the ways in which the actual shape varies from the idealized geometric model.

3 Model situations with equations and inequalities.

Include situations well described by a linear inequality in two variables or a system of linear inequalities defining a region in the plane.

4 Model situations with common functions.

Include situations well described by linear, quadratic or exponential functions; and situations that can be well described by inverse variation (( )= /). Include identifying a family of functions that models features of a problem, and identifying a particular function of that family and adjusting it to fit by changing parameters. Understand the recursive nature of situations modeled by linear and exponential functions.

5 Model situations using probability and statistics.

Include using simulations to model probabilistic situations; describing the shape of a distribution of values and summarizing a distribution with measures of center and variability; modeling a bivariate relationship using a trend line or a regression line.

6 Interpret the results of applying a model and compare models for a particular situation.

Include realizing that models seldom fit exactly and so there can be error; identifying simple sources of error and being careful not to over-interpret models. Include recognizing that there can be many models that relate to a situation, that they can capture different aspects of the situation, that they can be simpler or more complex, and that they can have a better or worse fit to the situation and the questions being asked.

Finally, some proposals

- Guiding principles
- Big Ideas
- A rational scope and sequence
- High Quality instruction
- Technology
- Assessments
- Projects
- Time

Guiding Principles 1

Scope principle: The Algebra curriculum must emphasize depth over breadth and a focus on the essential ideas and processes of algebra. While one often hears “less is more”, Singapore advocates “teach less, learn more.” We subscribe to a belief that traditional Algebra I has tried to teach too much to quickly with far too little depth.

Connections principle: The Algebra curriculum must focus on connections within mathematics and between mathematics and other disciplines. A key component of every lesson and unit is being explicit is one any one piece of mathematics relates to other pieces of mathematics. We should subscribe to the belief that too much algebra content has been fragmented and disconnected.

Context principle: Each of the Algebra curriculum units will consist of problem-based lessons built around real-world contexts, situations, and applications, often interdisciplinary in nature. We should subscribe to the belief that context often grounds and motivates learning.

Reasoning principle: The instruction and activities contained within the Algebra I curriculum will focus on “why?”, on explaining one’s thinking, and on justifying answers. We should subscribe to the belief that merely knowing “how” to get answers to exercises is not enough to demonstrate understanding.

Guiding Principles 2

Instructional principle: The Algebra curriculum will expect and support a set of research-affirmed instructional practices, including use of key questions, on-going cumulative review, attention to student thinking, addressing errors and common misconceptions, multiple representations of key ideas, alternative approaches to solving problems, and frequent opportunities to collaborate and communicate understanding. We should subscribe to the belief that **how** something is taught determines whether it is learned and how well.

Differentiation principle: The instructional design must include core experiences for all students as well as appropriate extensions and increased depth for some students and differentiation based on pacing and depth, while ensuring a set of assured experiences. We should subscribe to the belief that one-size-does-not-fit-all.

Assessment principle: The Algebra curriculum must include a balanced portfolio of strategically aligned and high quality assessments that will include opportunities for collaborative projects and student self-assessment. Benchmark tasks and formative assessment practices will be expected to identify areas for differentiation and student support. We should subscribe to the belief that what and how we assess Algebra I communicates what we value about algebra.

Guiding Principles 3

Technology principle: The Algebra curriculum must make full use of technological tools that can engage students in the learning of algebra and these tools must be ubiquitous throughout the curriculum, instruction, assessment, professional development, and marketing strategy components of the project. We should subscribe to the belief that the strategic use of technological tools is a critical component of increasing the productivity of instruction.

Essential Processes

Essential Processes (to be embedded in the content, instruction and assessment of the course):

- **Justifying** – students explain their reasoning and justify their answers
- **Modeling** – students use mathematical ideas to model real-world situations
- **Generalizing** – students regularly use mathematics to move from specific cases to the general relationships
- **Representing** – students use multiple representations of mathematical ideas
- **Predicting** – students regularly make estimates and predictions
- **Connecting** – students make connections between and among related mathematical ideas

2000 PSSM Algebra Standard

Instructional programs from pre-kindergarten through grade 12 should enable all students to:

- Understand patterns, relations, and functions;
- Represent and analyze mathematical situations and structures using algebraic symbols;
- Use mathematical models to represent and understand quantitative relationships;
- Analyze change in various contexts.

1989 5-8 Algebra Standard (Algebra for life)

- Understand the concepts of variable, expression, and equation;
- Represent situations and number patterns with tables, graphs, verbal rules and equations and explore the interrelationships of these representations;
- Analyze tables and graphs and identify properties and relationships;
- Develop confidence in solving linear equations;
- Investigate inequalities and non-linear equations informally;
- Apply algebraic methods to solve a variety of real-world and mathematical problems.

For example:
Sneaker Laces

- So? Describe a sneak lace.
- Relationships? Independent and dependent variables?
- Make a sign that relates the number of pairs of eyelets to the length.
- Mathematize the relationship.

For example:

What percent of the cars driving in front of the school are speeding?

Write a letter to the city council making the case for better enforcement.

Big Ideas 1

- **Functions** are the mathematical rules for taking input (independent variables) and producing output (dependent variables).
- Algebraic properties allow the generation of **equivalent forms** of most expressions and equations.
- Linear functions are **additive** (the dependent variable increases additively); exponential functions are **multiplicative** (the dependent variable increases multiplicatively).
- There is a direct relationship, for any function, among a **point** on a graph of the function, an **ordered pair** in a table of the function, and a **solution** to the symbolic form of the function.

Big Ideas 2

- Solving equations is an **undoing** process of equivalent equations based on **inverse** to isolate the variable, where addition undoes subtraction and vice verse, multiplication undoes division, square rooting undoes squaring.
- The graphs of functions can be visualized and predicted based on **transformations** of a parent function (all linear functions are transformations of the parent function or line $y = x$; all quadratic functions are transformations of the parent function $y = x^2$).
- The graph of the **inverse** of a function is its reflection across the line $y = x$.
- Just as subtraction undoes, or is the inverse of, addition, the square root is the inverse of squaring and finding a log is the inverse of raising to a power.

A proposal for Algebra 1

Unit 1: Patterns

Unit 2: Equations

Unit 3: Linear functional situations

Unit 4: Representing functional situations

Unit 5: Direct and indirect variation

Unit 6: Data

Unit 7: Systems of equations

Unit 8: Exponential functions

Unit 9: Linear programming

Traditional Algebra 1

Unit 1: Expressions and Equations

Chapter 1	The Language of Algebra
Chapter 2	Real Numbers
Chapter 3	Solving Linear Equations

Unit 2: Linear Functions

Chapter 4	Graphing Relations and Functions
Chapter 5	Analyzing Linear Equations
Chapter 6	Solving Linear Inequalities
Chapter 7	Solving Systems of Linear Equation and Inequalities

Unit 3: Polynomials and Nonlinear Functions

Chapter 8	Polynomials
Chapter 9	Factoring
Chapter 10	Quadratic and Exponential Functions

Unit 4: Radical and Rational Functions

Chapter 11	Radical Expressions and Triangles
Chapter 12	Rational Expressions and Equations

Unit 5: Data Analysis

Chapter 13	Statistics
Chapter 14	Probability

A proposal for Algebra 2

Unit 1: Review and reinforce big ideas and key skills of Algebra 1

Unit 2: Quadratic functions

Unit 3: Polynomials and polynomial functions

Unit 4: Patterns, series and recursion

Unit 5: Exponential and logarithmic functions

Unit 6: Rational and radical functions

Unit 7: Probability and statistics

Unit 8: Optimization, graph theory and topics in discrete mathematics

Traditional Algebra 2

Chapter 1:	Analyzing Equations and Inequalities
Chapter 2:	Graphing Linear Relations and Functions
Chapter 3:	Solving Systems of Linear Equations and Inequalities
Chapter 4:	Using Matrices
Chapter 5:	Exploring Polynomials and Radical Expression
Chapter 6:	Exploring Quadratic Functions and Inequalities
Chapter 7:	Analyzing Conic Sections
Chapter 8:	Exploring Polynomial Functions
Chapter 9:	Exploring Rational Expressions
Chapter 10:	Exploring Exponential and Logarithmic Functions
Chapter 11:	Investigating Sequences and Series
Chapter 12:	Investigating Discrete Mathematics and Probability
Chapter 13:	Exploring Trigonometric Functions
Chapter 14:	Using Trigonometric Graphs and Identities

A proposal for Algebra 1

Unit 1: Patterns – introduces many of the big ideas, sets instructional routines, uses tile patterns to display patterns graphically.

- The student will represent a given tile pattern with words, in a table, with a graph or with an equation and identity how they are equivalent.
- The student will model algebraic expressions in a variety of ways (for example, using algebra tiles, sketches/diagrams, bar models).
- The student will use real number properties (identity, inverse, commutative, associative, distributive) to simplify and evaluate algebraic expressions and to write equivalent expressions.
- The student will create and differentiate iterative and explicit rules for patterns.
- The student will recognize, describe, and extend arithmetic sequences; determine a specific term of a sequence when given an explicit formula; write an explicit rule for the nth term of an arithmetic sequence; and write a recursive rule for the nth term of an arithmetic sequence.

A proposal for Algebra 1

Unit 2: Equations

- The student will model algebraic equations in a variety of ways (for example, using algebra tiles, sketches/diagrams, bar models) and will identify and create equivalent equations.
- Given a mathematical or real-world situation, the student will define a variable, write an equation or an inequality, solve the equation or inequality, and interpret the solution.
- The student will graph the solutions of equations and inequalities on a number line.
- The student will use real number properties (identity, inverse, commutative, associative, distributive) to justify steps in solving equations and inequalities.

A proposal for Algebra 1

Unit 3: Linear functional situations – introduces the concept of function via activities that relate two variables

- The student will recognize and apply real-world functions in a variety of representations and translate among verbal, tabular, graphic, and algebraic representations of functions.
- The student will recognize an example of a function; identify the role of independent and dependent variables in a function; determine the domain and range of a linear function; find the slope and intercepts of a linear function; and use function notation to evaluate a function for a specified value.

A proposal for Algebra 1

Unit 4: Representing function situations

- The student will recognize functions in a variety of representations and a variety of contexts and translate among verbal, tabular, graphic, and algebraic representations of functions (including the use of function notation, f(*n*)).
- The student will describe how the aspects of the function such as the dependent and independent variables and slope and y-intercept are reflected in the different representations.
- The student will explain how the change in one variable affects the change in another variable.
- The student will understand that the slope of a line represents a constant rate of change.
- The student will show how changes in parameters affect the graph of the function.

A proposal for Algebra 1

Unit 5: Direct and indirect variation

- The student will recognize that when the ratio between two varying quantities is invariant, the two quantities are said to be directly proportional, and when the product of two varying quantities is invariant, the two quantities are said to be inversely proportional.
- The student will use proportional relationships and proportional reasoning to solve real-world problems.
- The student will distinguish directly proportional relationships ($y/x = k$ or $y = kx$) from other relationships, including inverse proportionality ($xy = k$ or $y = k/x$).
- The student will recognize that $y=kx$ represents a proportional relationship and that when $b≠0$, $y=kx+b$ does not represent a proportional relationship.
- The student will recognize that the graph of a proportional relationship is a line that passes through the point (0,0).
- The student will translate among verbal, tabular, graphical, and algebraic representations of direct and inverse variation.
- The student will apply direct and inverse variation to solve real-world and mathematical problems.

A proposal for Algebra 1

Unit 6: Data (scatterplots, correlation, lines of best fit

- The student will construct, interpret and draw and justify conclusions from scatter plots.
- The student will describe the advantages and disadvantages of using scatter plots to represent data.
- The student will describe relationships in data represented in scatter plots (linear, non-linear, positive correlation, negative correlation, no correlation).
- The student will find an equation that represents linear trend, when it is appropriate, for real-world data and use the equation, table, or graph to make predictions and solve real-world problems.

A proposal for Algebra 1

Unit 7: Systems of equations

- The student will understand that a system of simultaneous linear equations in two unknowns contains two distinct linear equations with two unknowns and the solution to the system is the point (x, y) that makes both equations true.
- Given a mathematical or real-world situation, the student will define the variables and write an appropriate system of linear equations.
- The student will use tables and graphs to find and interpret solutions to systems of equations in mathematical and real-world contexts.
- The student will describe the characteristics of the slopes of parallel and perpendicular lines in terms of slope.

Unit 8: Exponential Functions (multiplicative change vs. additive change)

- The student will compare and contrast linear and exponential models.
- The student will describe the role of the parameters in context
- The student will write and use models for exponential growth and decay

Unit 9: Linear Inequalities and Linear Programming (A capstone unit)

Instruction 1

- **Effective mathematics instruction is thoughtfully planned.**
- **The heart of effective mathematics instruction is an emphasis on problem solving, reasoning, and sense-making.**
- **Effective mathematics instruction balances and blends conceptual understanding and procedural skills.**

Instruction 2

- **Effective mathematics instruction relies on alternative approaches and multiple representations.**
- **Effective mathematics instruction uses contexts and connections to engage students and increase the relevance of what they are learning.**
- **Effective mathematics instruction provides frequent opportunities for students to communicate their reasoning and engage in productive discourse.**

Instruction 3

- Effective mathematics instruction incorporates ongoing cumulative review.
- Effective mathematics instruction employs technology to enhance learning.
- Effective mathematics instruction maximizes time on task.
- Effective mathematics instruction uses multiple forms of assessment and uses the results of this assessment to adjust instruction.

Instruction 4

- **Effective mathematics integrates the characteristics of this vision to ensure student mastery of grade-level standards.**
- **Effective mathematics teachers reflect on their teaching, individually and collaboratively, and make revisions to enhance student learning.**

Long Reach HS

Howard County (MD) recognized that there were a significant number of 9th graders who were not being successful in Algebra 1. To address this problem, the county designed Algebra Seminar for approximately 20% of the 9th grade class in each high school. These are students who are deemed unlikely to be able to pass the state test if they are enrolled in a typical one-period Algebra I class. Algebra Seminar classes are:

- **Team-taught with a math and a special education teacher;**
- **Systematically planned as a back-to-back double period;**
- **Capped at 18 students;**
- **Supported with a common planning period made possible by Algebra Seminar teachers limited to four teaching periods;**
- **Supported with focused professional development;**
- **Using Holt Algebra I, Carnegie Algebra Tutor, and a broad array of other print and non-print resources;**
- **Notable for the variety of materials and resources used (including Smart Board, graphing calculators, laptop computers, response clickers, Versatiles, etc.);**
- **Enriched by a wide variety of highly effectively instructional practices (including effective questioning, asking for explanations, focusing of different representations and multiple approaches); and**
- **Supported by county-wide on-line lesson plans that teachers use to initiate their planning.**

Jo Boaler's Work

Action	Typical HS	Railside HS
Lecture	21%	4%
Questioning	15%	9%
Individual Work Practicing	48%	
Group Work		72%
Student Presenation	0.2%	9%

Jo Boaler's Work

- **Typical Class:**
  - 2.5 minutes/problem
  - 24 problems/class
- **Railside HS class:**
  - 5.7 minutes/problem
  - 16 problems/90 minute period

Jo Boaler's Work
Multidimensional classes

"In many classrooms there is one practice that is valued above all others – that of executing procedures (correctly and quickly). The narrowness by which success is judged means that some students rise to the top of classes, gaining good grades and teacher praise, while other sink to the bottom with most students knowing where they are in the hierarchy created. Such classrooms are unidimensional."

Jo Boaler's Work
Multidimensional classes

"At Railside the teachers created multidimensional classes by valuing many dimensions of mathematical work. This was achieved, in part, by having more open problems that students could solve in different ways. The teachers valued different methods and solution paths and this enabled more students to contribute ideas and feel valued."

When there are many ways to be successful, many more students are successful.

"When we interviewed the students and asked them "what does it take to be successful in mathematics class?" they offered many different practices such as: asking good questions, rephrasing problems, explaining well, being logical, justifying work, considering answers…

When we asked students in “traditional” classes what they needed to do in order to be successful they talked in much more narrow ways, usually saying that they needed to concentrate, and pay careful attention.”

So….

While we acknowledge the range and depth of the problems we face,

It should be comforting to see the availability and potential of solutions to these problems….

Now go forth and start shifting YOUR school's Algebra 1 and 2 courses to better serve our students, our society and our future.

Thank you.

SLeinwand@air.org

Example 2:

$$F = 4\,(S - 65) + 10$$

Find F when S = 81

Vs.

First I saw the blinking lights… then the officer informed me that:

The speeding fine here in Vermont is $4 for every mile per hour over the 65 mph limit plus a $10 handling fee.

Connecticut: $F = 10\ (S - 55) + 40$

Maximum speeding fine: $350

- **Describe the fine in words**
- **At what speed does it no longer matter?**
- **At 80 mph how much better off would you be in VT than in CT?**
- **Use a graph to show this difference**

Example 3:

Solve for x: $16 \times .75^x < 1$

Vs.

You ingest 16 mg of a controlled substance at 8 a.m. Your body metabolizes 25% of the substance every hour. Will you pass a 4 p.m. drug test that requires a level of less than 1 mg? At what time could you first pass the test?

My Aunt Irma

My Aunt Irma, who is so overweight she must sit in two chairs, finally decides to go on a diet when she reaching 300 pounds. She manages to lose 2 ¼ pounds each week, except for every fourth week when she goes on a binge and gains 1 ¼ pounds instead of losing any.

How long will it be before she loses 40 pounds?

How much will she lose in a year?

How long will be before she disappears entirely?

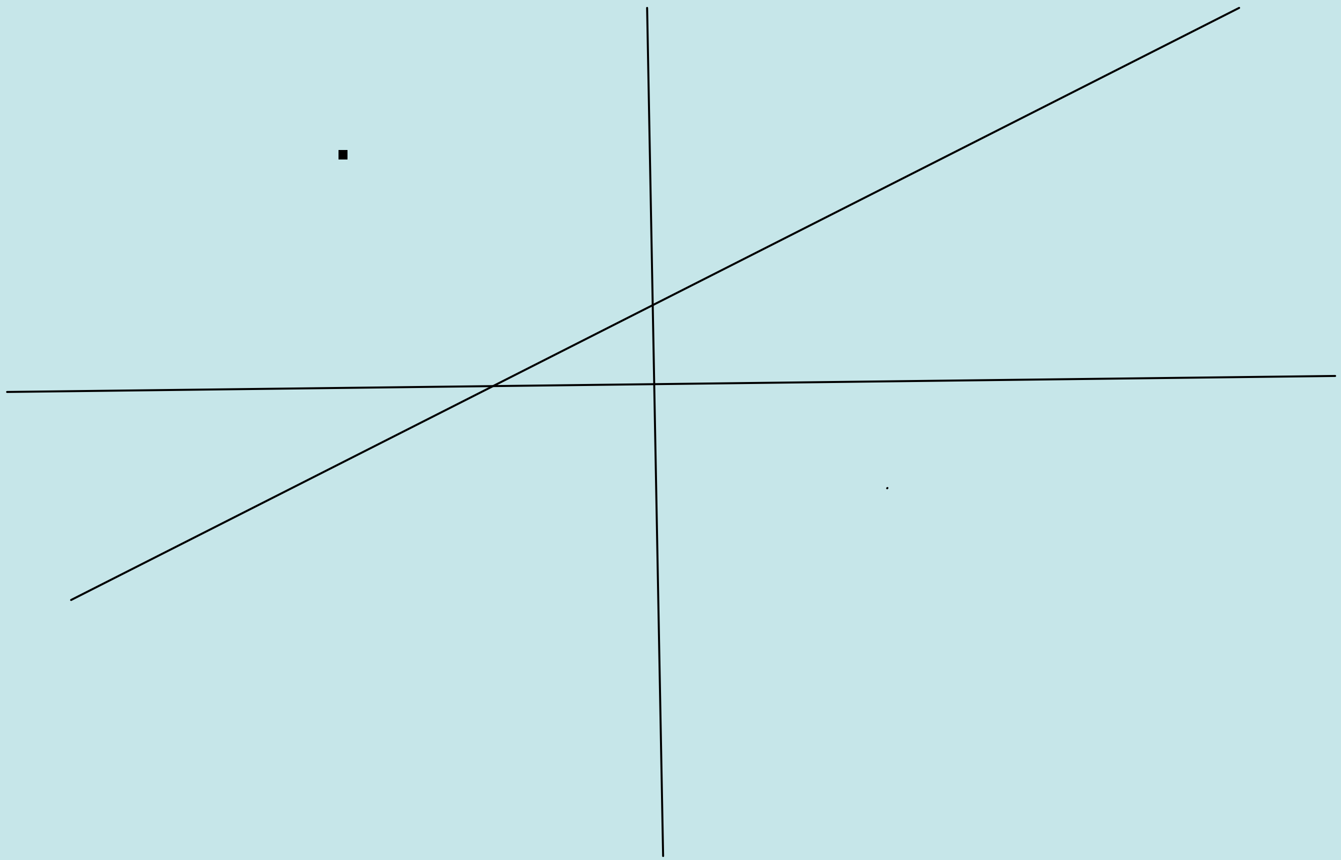
0	210	158	113
2	202	154	108
4	196	150	105

Weeks	Tom	Jamie	Roni
0	210	158	113
2	202	154	108
4	196	150	105

A weighty matter

- What is each person’s weight after 6 weeks on the diet?
- How is doing best so far?
- Argue that Tom is best. Jaime is best. Roni is best.
- Express weight P of each person after w weeks.

Tell me what you see.



Tell me what you see.

$$f(x) = x^2 + 3x - 5$$

So how do we know if anything has been learned?

The fine for speeding on the highways in most states is a function of the speed of the car. In one state, the fine can be determined using the formula:
$F = 10 (S – 65) + 32$
Where F is the fine in dollars and S is the speed your car was going in miles per hour.

a. What would the speeding fine be if you were caught going 88 miles per hour in this state? Show how you arrived at your answer.

b. Suppose you received a speeded fine for $112. How fast were you going? Show or explain you how arrived at your answer.

c. Explain what the 10 represents in the formula.

d. Explain what the 32 represents in the formula.

e. Why is it unlikely for someone to receive a speeding fine of $52 in this state?

At the Downtown Diner, a server is paid $40 per day plus 75% of the tips. The cook earns $75 per day plus the remaining 25% of the tips.

a. How much does each person earn on a day when the tips total $46? Show how you arrived at your answers.

b. For what amount of tips would the server and the cook earn the same amount of money? Show how you arrived at your answer.

c. On the grid below, sketch a double line graph that represents the server's earnings and the cook's earnings at the tips range from $0 to $200.

During a Saturday clean-up, 12 volunteers collected 900 pounds of garbage.

a. If 320 people volunteered the following weekend and worked at the same rate, how many pounds of garbage would you expect to be collected? Show how you arrived at your answer.

b. The Sanitation Department estimates that there is approximately 15,000 pounds of garbage that still needs to be collected. About how many volunteers would you need to recruit to volunteer for the Saturday clean-up effort? Show how you arrived at your answer.

Aida's bank account starts at $1000 and grows by $30 each week. Jerome's bank account starts at $600 and grows by 10% each week.

After how many weeks will Jerome's bank account first have more money than Aida's bank account? Show your work and explain with a table, a graph or equation how your arrived at your answer.

A science class is planning a field trip to a local museum. The rental cost of a bus for the trip is $300 and will be shared equally among the students who will go on the trip.

a. Sketch a graph that shows the cost of the bus for each student as a function of the number of students going on the trip.

b. If N is the number of students who will be going on the trip and C is the cost in dollars for each student, write an equation for C in terms of N.

McChicken King sells packages of 5 chicken nuggets for $1.59 and packages of 12 chicken nuggets for $2.79. Assume that there is a linear relationship between the number of nuggets and the cost.

a. Sketch a graph that shows the price of packages of nuggets from 2 to 30.

b. Write an equation that expresses the cost C of a package of nuggets as a function of N, the number of nuggets in the package.

c. McChicken King decides to start selling packages of 20 nuggets. What would you expect the cost of these packages to be? Explain how you arrived at your answer.

Assumptions

Assumptions about a revised Algebra 1 and 2 scope and sequence:

- Topics such as matrices, series and sequences, trigonometry and conic sections have all been covered or considered for coverage in Algebra II courses. While important, none of these topics is deemed more important than the skills and concepts listed below and all can be effectively addressed in a Pre-calculus course that follows Algebra II.
- Given the focus on functions and the connections between symbolic, tabular and graphical representations, we assume that a graphics calculator will be available to every student and effectively used to enhance the learning of the mathematical content of Algebra II and to better apply the skills and concepts of the course.
- The fundamental purpose of Algebra II is to continue to develop and deepen students' understanding of functions and their applications. As such, Algebra II builds on and reinforces the skills and concepts of Algebra I, prepares students for the skills and concepts of pre-Calculus and Calculus, prepares students for the mathematics required by a statistics course, and prepares students to successfully take the SAT and ACT college admissions examinations.