Polygon Potpourri

Investigations in Geometry

CMC-S Palm Springs
November 2, 2013
with Michael Serra
Donut Polygons

A *donut polygon* (also called a polygon gasket or pierced polygon) is the union of two polygons, one in the interior of the other. If the outer polygon has \( n \) sides and the polygon in its interior has \( m \) sides then the donut polygon is symbolized by \( nP_m \). The donut polygon below left is a \( 4P_6 \) donut polygon. It is also a convex donut polygon since both polygons are convex. If one or both polygons are concave then the polygon is a concave donut polygon. The \( 8P_4 \) and \( 6P_6 \) donut polygons below right are concave donut polygons.

**Investigation 1.**
What is the sum of the measures of the interior angles of a convex \( nP_m \) donut polygon? By interior angle of a donut polygon we mean an angle measured in the region between the two polygons. Second, what is the relationship between \( nP_m \) and \( mP_n \)?

**Investigation 2.**
What is the sum of the measures of the interior angles of a concave \( nP_m \) donut polygon (where each polygon has at most one dent)?

**Investigation 3.**
What if the concave polygons have two “dents”? What is the sum of the measures of the interior angles of a concave \( nP_m \) donut polygon where at least one or both polygons have two dents?
Star Polygons

**Star Polygons**
Creating a 5-point star polygon by connecting every second point

Step 1. Make 5 points
Step 2. Connect 2 points with a segment skipping a point
Step 3. Continue connecting every other point.

Creating a 7-point star polygon by connecting every third point

Step 1. Make 7 points
Step 2. Connect 2 points with a segment skipping two points
Step 3. Continue connecting every third point.

**Special Case Star Polygons**
Creating a 6-point star polygon by connecting every fourth point

Step 1. Make 6 points
Step 2. Connect every fourth point.
Step 3. Pick another point and repeat.

Creating a 6-point star polygon by connecting every third point

Step 1. Make 6 points
Step 2. Connect every third point.
Step 3. Pick another point and repeat.

<table>
<thead>
<tr>
<th>Number of points</th>
<th>Connecting every first point</th>
<th>Connecting every second point</th>
<th>Connecting every third point</th>
<th>Connecting every fourth point</th>
<th>Connecting every fifth point</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
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<td>180°</td>
<td>180°</td>
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<td>180°</td>
</tr>
<tr>
<td>4</td>
<td>360°</td>
<td>0</td>
<td>360°</td>
<td>180°</td>
<td>180°</td>
</tr>
<tr>
<td>5</td>
<td>540°</td>
<td>180°</td>
<td>180°</td>
<td>180°</td>
<td>180°</td>
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<tr>
<td>6</td>
<td>720°</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>(5)180°</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>(6)180°</td>
<td></td>
<td></td>
<td></td>
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<td>(8)180°</td>
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<tr>
<td>12</td>
<td>(10)180°</td>
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<td></td>
</tr>
<tr>
<td>( n )</td>
<td>( (n - 2)180° )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**Concave Polygons**

**Investigation #1.** What is the sum of the measures of the interior angles of a concave polygon?

Conjecture:

Proof:

**Investigation #2.** What is the sum of the measures of the exterior angles of a concave polygon?

Definition of an exterior angle?

If we consider the measure of the exterior angle at the “dent” as the angle measured on the exterior (for example x would be the measure of the exterior at the dent for the dented polygon on the right) then what is the sum of the measures of all the exterior angles?

Conjecture:

If we consider the measure of the exterior angle at the “dent” as measured from a side extended to the next side (for example x would be the measure of the exterior at the dent for the polygon on the right) then what is the sum of the measures of all the exterior angles?

Conjecture:
**Cyclic Polygons**

A *cyclic polygon* is a polygon that can be inscribed in a circle.

**Investigation #1.** Are all triangles cyclic polygons? Prove or disprove. Are all squares, rectangles and isosceles trapezoids cyclic polygons? Prove or disprove. Are there any other parallelograms that are cyclic polygons? Prove or disprove.

**Investigation #2.** Investigate cyclic $n$-gons where $n$ is even. What do you notice about alternating angles? Make a conjecture and prove.
Cryptography and Grille Cipher Polygons

Cryptography is the science of writing secret messages. Cryptography dates back to about 1900 B.C., when an Egyptian scribe used special hieroglyphs to disguise a message.

According to legend, Julius Caesar used a simple letter substitution method to send secret messages to his military leaders. Caesar replaced each letter of his message with the letter that followed it by three positions in the Roman alphabet. For example, in English, the letter “a” would be replaced by “D”, the letter “t” would be replaced by “W”, and “x” would be replaced by “A.” Thus, the English phrase “strike at dawn” would be transformed to "VWULNH DW GDZQ.” This technique of sending secret messages, no matter the size or direction of the shift used for the substitution, is called a Caesar cipher.

The process of creating a secret message is called encryption; it is what the sender does to hide the message. The process of translating the secret message is called decryption; it is what the receiver does to reveal the original message. The original message is known as the plaintext and the encrypted plaintext is called the ciphertext.

Transposition Ciphers

In transposition ciphers the letters remain the same but the arrangement has changed (transposed) according to a system. The secret word “treasure” rewritten as “steareur” would be an example of a transposition. People who are good with anagrams might prefer to encrypt their messages by transposition.

Grille Cipher

One transposition cipher, the grille cipher, positions the ciphertext in a square grid. The cipher key is a square grid with cut-out squares called the grille. To decipher the message you place the grille over the ciphertext square grid and read off the letters showing through the cut-out squares as they appear, left to right top to bottom. Next, you rotate the grille 90° clockwise and read the letters again. Rotate the grille 180° and repeat. Finally, rotate the grille 270° and repeat.

For example, the message “dig here 3 feet down” when encrypted looks like the 4×4 grid to the near right. The grille is shown to the far right.

When the grille is placed over the ciphertext the beginning of the message is revealed “digh”. When the grille is rotated 90° clockwise the message continues when “ere3” is revealed.

Let’s call this cipher key the GC.1.6.8.14. (grille cipher with holes at squares 1, 6, 8, and 14.) The four positions of the grille are shown below revealing the message.

<table>
<thead>
<tr>
<th>0° rotation</th>
<th>90° rotation</th>
<th>180° rotation</th>
<th>270° rotation</th>
</tr>
</thead>
<tbody>
<tr>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
</tr>
<tr>
<td>i</td>
<td>d</td>
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<td>e</td>
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<td>g</td>
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<td>o</td>
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<tr>
<td>h</td>
<td>r</td>
<td>e</td>
<td>n</td>
</tr>
<tr>
<td>3</td>
<td>e</td>
<td>e</td>
<td>t</td>
</tr>
</tbody>
</table>

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The key for the grille cipher shown to the right is GC.1.8.10.17.20.23 (grille cipher with holes at squares 1, 8, 10, 17, 20, and 23). Copy the 5×5 grille to the right onto patty paper or tracing paper. Don’t copy the letters. Shade in all the squares except 1, 8, 10, 17, 20, and 23.

20. Use this sheet to decipher the hidden message.

Notice that six squares are cut out (1, 8, 10, 17, 20, 23). Since each square gets rotated (each occupying four positions) the message is 24 letters or numbers. The holes never land on the center square and thus, the center square “x” is not part of the message.

Plaintext: ________________________________________________

Excerpt from Notes Document

29. Isla de Oro

GC.H.
2.12.15.19.25.28.33.4
7.50
Suggested Reading in Geometry

• Discovering Geometry 4th edition, Serra, Kendall Hunt Publishing

• Patty Paper Geometry, Serra, Playing It Smart 1994

• What’s Wrong With This Picture? , Serra, Playing It Smart 2003

• Smart Moves Developing Mathematical Reasoning with Games and Puzzles, Serra, Playing It Smart 2011

• Pirate Math Developing Mathematical Reasoning with Games and Puzzles, Serra, Playing It Smart (To be released early 2014)

• Polyominoes, Solomon Golomb, Charles Scribner’s Sons

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